Notes on the possibilities and limitations of using metrical tools in mechanical modeling of media

G Lámer

University of Debrecen, Faculty of Engineering, Department of Engineering Management and Enterprise. 4028 Hungary, Debrecen, Ótemető utca 2-4.

glamer@eng.unideb.hu

Abstract. Changes in the size and shape of a continuous body can be described by metrical tools. The change in size is experimentally associated with relative elongation, the change of shape with the (local) angle change. Within the body, the concept of size (line length) and angle (angle between two lines) is based on the scalar product; the metric tensor is also needed to describe these relationships. When determining the relationship between size and size change as well as angle and angle change, it should be taken into account that, although both length and angle are interpreted by scalar product of metric tensor components, but in the context of relative length change there are fractional functions of radical expressions, in the context of angular change, there are fractional functions of radical expressions and partly inverse angle functions. The tensor cannot be formed directly from these two functions, but only by series expansion, and then there is a limit to the theoretical and numerical application of the introduced strain tensor: that the whole function can be approached well with the first few members of series expansion. This allows only a small strain (small relative elongation) to be described. This limit and the limitation of the use of topological tools (that is, the significant change of size and shape is not a deformation but a rearrangement) are consistent with each with other.

1. Introduction

Generally, the symmetrical linear expression constructed from the gradient of the displacement vector of the body is used for the local description of continuous media, $\boldsymbol{\varepsilon}_L = (\nabla\mathbf{u} + (\nabla\mathbf{u})^*/2,$ or an expression including first degree and quadratic elements, $\boldsymbol{\varepsilon}_{NL} = (\nabla\mathbf{u} + (\nabla\mathbf{u})^* + \nabla\mathbf{u}(\nabla\mathbf{u})^*)/2.$ Generally, the ε_L expression is associated with small strain, while the ε_{NL} expression is associated with large strain. It is known from the literature [1,2], that nonlinear relationships are necessary in case of movement without strain for the strains to "disappear". According to this, it is not the linear but the nonlinear expressions that belong not even the small, but rather to the zero strains. Based on the principle of continuity - the expression suitable for describing small strains is closer to zero than the expression describing large strain - this is incomprehensible.

It is also known from the literature (see, for example, the above references) that the relationships explained above from the metric tensor to describe the physical strains are not given "automatically", but can be interpreted mathematically in relation to relative elongations and relative angular changes such that the root function and the angle function are associated with the first one or two elements of its power series. This assumes that the absolute value of the expressions tested is small (negligible to compare with 1 (one)).

These examinations are fundamentally based on previous research [3,4,5].

2. Differential geometric description of the continuous medium

2.1. Differential geometric background

On the geometric background see e.g. [6,7], and [8]; theory of elasticity [9], and [10,2]. In this subsection, the overview is based on results from literature; our aim is to list relationships to support further arguments. In addition to the list, we also focus on presenting step by step how each geometric concept is built upon each other.

The coordinate system $\{q^i\}$ should be given in space.

Mark the position vector before the deformation with **r**, and **R** after the deformation. The following relationships are written only for the state before the deformation. They apply to analog relationships following the change in strain.

First we look at the quantities describing the geometry.

Basis vectors in position vector **r**: $\mathbf{r}_i = \frac{\partial \mathbf{r}}{\partial q^i}$.

Components of metric tensor: $g_{ij} = \langle \mathbf{r}_i, \mathbf{r}_j \rangle$.

Partial differentials of basis vectors: $\mathbf{r}_i = \frac{\partial \mathbf{r}_i}{\partial q^j} \left(= \frac{\partial \mathbf{r}_j}{\partial q^i} \right) = \Gamma^k_{ij} \mathbf{r}_k \left(= \Gamma^k_{ji} \mathbf{r}_k \right)$.

The affine coefficients (Christoffel-type symbols): $\Gamma^k_{ij} = g^{ks} \langle \mathbf{r}_{ij}, \mathbf{r}_s \rangle = \frac{1}{2} g^{ks} \left(\frac{\partial g_{is}}{\partial q^j} + \frac{\partial g_{js}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^s} \right)$.

Curvature tensor describing the curvature of space: $r^i{}_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial q^k} - \frac{\partial \Gamma^i_{jk}}{\partial q^l} + \Gamma^i_{sk} \Gamma^s_{jl} - \Gamma^i_{sl} \Gamma^s_{jk}$.

Then we look at those geometric terms, which can be expressed with the amounts above.

Scalar product of vectors **a** and **b**: $\langle \mathbf{a}, \mathbf{b} \rangle = g_{ij} a^i b^j$.

The (cosine of) angle between vectors **a** and **b**: $\cos(\mathbf{a}, \mathbf{b}) = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\sqrt{\langle \mathbf{a}, \mathbf{a} \rangle \langle \mathbf{b}, \mathbf{b} \rangle}}$.

The length of the regular arc between two points of the space: $s = \int_{t_0}^{t_1} ds = \int_{t_0}^{t_1} \sqrt{\left| g_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt} \right|} dt$.

The absolute differentials (presented for one contravariant component): $\nabla_j a^i = \frac{da^i}{dq^j} + \Gamma^i_{kj} a^k$.

Parallel displacement and the difference resulting from interchanging absolute differentials can be described with the curvature tensor.

2.2. Relationships among the geometric objects before and after deformation

See relationships in general e.g. [9,10,2].

The difference of position vectors is the displacement vector: $\mathbf{u} = \mathbf{R} - \mathbf{r}$.

The difference of metric tensors in the measure tensor of strain: $\boldsymbol{\gamma} = \mathbf{G} - \mathbf{g}$; different measure tensors of strain are usually interpreted, generally half of those given here. We will come back to this later.

The change of affine connexion coefficient: $\lambda^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ij}$. Generally, no geometric content is associated with it. In case of small strains *and* small displacements, this change is negligible, in case of small strains *and* large displacements cannot be neglected (see e.g. [4,5]).

Change of curvature tensor: $\boldsymbol{\rho} = \mathbf{R} - \mathbf{r}$, that is $\rho^i{}_{jkl} = R^i{}_{jkl} - r^i{}_{jkl}$; give the compatibility equations (if it is made to equal zero).

To construct the theory, it is necessary to express the change of different geometric objects with the gradient tensor of the displacement vector. Here, we only give the relationship for one geometric object, the measure tensor of strain: $\boldsymbol{\gamma} = \nabla\mathbf{u} + (\nabla\mathbf{u})^* + (\nabla\mathbf{u})(\nabla\mathbf{u})^*$. We do not use the other relationships in this study, for relationships see [3,4,5].

3. Relationship of displacement and rotation during the strain of continuous medium

The change of continuous medium due to external effect is hereinafter referred to as mapping. The mapping of the continuous medium locally is usually split into the "product" of rotation and strain with polar division (see e.g. [11,12,13], or [9,10,]).

The mapping of a continuous medium is given with the displacement vector: $\mathbf{u} = \mathbf{R} - \mathbf{r}$. The displacement vector is the kinematic variable of the continuous medium.

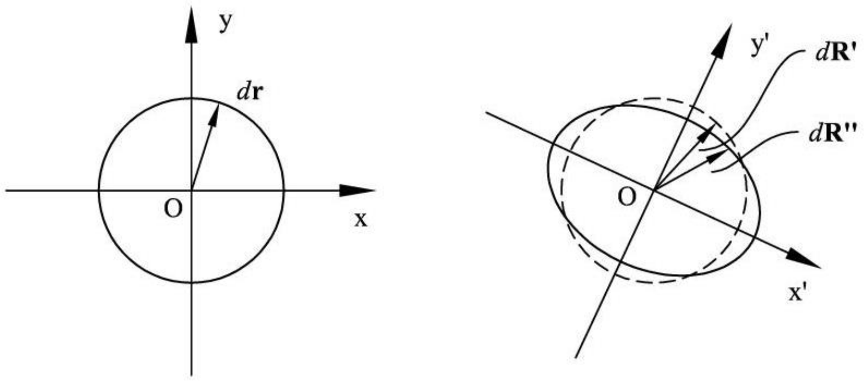

Figure 1. Rotation of a direction vector and the neighbourhood of a point:
$d\mathbf{R}' = \boldsymbol{\Omega}\cdot d\mathbf{r}$; $d\mathbf{R}'' = (\nabla\mathbf{R})^* d\mathbf{r}$

The tensor $\boldsymbol{\Omega}$ obtained by polar division ($\nabla\mathbf{R} = \boldsymbol{\Omega}\mathcal{A}$) of gradient tensor $\nabla\mathbf{R}$ of mapping $\mathbf{u}$ gives the rotation of the neighborhood of the point, $d\mathbf{R}' = \boldsymbol{\Omega}\, d\mathbf{r}$ (see figure 1).

The gradient tensor $\nabla\mathbf{R}$ of mapping $\mathbf{u}$ gives the rotation of a direction vector marked with $d\mathbf{r}$ of position vector $\mathbf{r}$, and the same time the change in its length: $d\mathbf{R}'' = (\nabla\mathbf{R})^* d\mathbf{r}$ (see figure 1).

Based on the above, the following conclusions can be made. In the continuous medium we can interpret two rotations. One is the rotation of the direction, and the rotation of the neighborhood of a point. Both depend on the kinematic variable of the continuous medium, the displacement vector **u**.

The individual decomposition can be expressed by the gradient tensor of the displacement vector **u**.

In the linear case, for this reason, the subscript L is used, we get the following terms:

$$(\nabla\mathbf{R})^* = \mathbf{g} + (\nabla\mathbf{r})^* = \mathcal{A}_L \circ \boldsymbol{\Omega}_L = \boldsymbol{\Omega}_L \circ \mathcal{A}_L = \mathbf{g} + \boldsymbol{\varepsilon}_L + \boldsymbol{\omega}_L, \tag{1}$$

where

$$\mathcal{A}_L = \mathbf{g} + \boldsymbol{\varepsilon}_L, \tag{2}$$

$$\boldsymbol{\Omega}_L = \mathbf{g} + \boldsymbol{\omega}_L, \tag{3}$$

$$\boldsymbol{\varepsilon}_L = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^*), \tag{4}$$

$$\boldsymbol{\omega}_L = \frac{1}{2}((\nabla\mathbf{u})^* - \nabla\mathbf{u}). \tag{5}$$

In the linear case, both the right and left rotation tensors ($\mathbf{\Omega}_L$) as well as the right and left tensors connected with the strain ($\mathcal{A}_L$) coincide; the $\mathbf{\Omega}_L$ and $\mathcal{A}_L$ quantities can be interchanged with error negligible to compare with 1 (one).

Note. It is known from linear algebra that $\boldsymbol{\omega}_L = ((\nabla\mathbf{u})^* - \nabla\mathbf{u})/2$ tensor quantity does not describe rotation but interprets vector product; the expression $\mathbf{\Omega}_L = \mathbf{g} + \boldsymbol{\omega}_L$ itself rotates, if $\boldsymbol{\omega}_L$ is small compare with $\mathbf{g}$ [7,14].

In the quadratic case, for this reason, the subscript *KV* is used, as well as subscript "*le*" refers to the left, subscript "*ri*" refers to the right decomposition, we get the following terms:

$$(\nabla\mathbf{R})^* = \mathcal{A}_{le\,KV} \circ \mathbf{\Omega}_{KV} = \mathbf{\Omega}_{KV} \circ \mathcal{A}_{ri\,KV} = \mathbf{g} + (\nabla\mathbf{r})^*, \tag{6}$$

where

$$\mathcal{A}_{le\,KV} = \mathbf{g} + \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^*) - \frac{1}{8}((\nabla\mathbf{u})(\nabla\mathbf{u}) + (\nabla\mathbf{u})^*(\nabla\mathbf{u})^*) - \frac{1}{8}(\nabla\mathbf{u})(\nabla\mathbf{u})^* + \frac{3}{8}(\nabla\mathbf{u})^*\nabla\mathbf{u}, \tag{7}$$

$$\mathcal{A}_{ri\,KV} = \mathbf{g} + \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^*) - \frac{1}{8}((\nabla\mathbf{u})(\nabla\mathbf{u}) + (\nabla\mathbf{u})^*(\nabla\mathbf{u})^*) + \frac{3}{8}(\nabla\mathbf{u})(\nabla\mathbf{u})^* - \frac{1}{8}(\nabla\mathbf{u})^*\nabla\mathbf{u}, \tag{8}$$

$$\mathbf{\Omega}_{KV} = \mathbf{g} + \frac{1}{2}((\nabla\mathbf{u})^* - \nabla\mathbf{u}) + \frac{3}{8}(\nabla\mathbf{u})(\nabla\mathbf{u}) - \frac{1}{8}(\nabla\mathbf{u})^*(\nabla\mathbf{u})^*) - \frac{1}{8}(\nabla\mathbf{u})(\nabla\mathbf{u})^* - \frac{1}{8}(\nabla\mathbf{u})^*\nabla\mathbf{u}. \tag{9}$$

The left and right decompositions do not match. The fact that the tensors connected with deformations in the left and right decomposition are not the same raises the possibility that nature chooses between the left and right decompositions: among the two possibilities of deformation, experimental results should be distinguished. The author is not aware of any theoretical foundations and experimental results. The components of the strain tensor will be determined using components of metrical tensors (in deformed and non-deformed states), so there is no need to use the tensors $\mathcal{A}_{le\,KV}, \mathcal{A}_{ri\,KV}$, see Section 4.

4. Interpretation of strain in a continuous medium

4.1. Experimental interpretation of strain

The strain was derived from experimental experience. Two types of strain are usually distinguished [13]: specific length change, independent of the length of the prismatic rod (specimen), and the specific rotation of each cross-section of the circular cylinder (specimen) independent of the length of the specimen.

The length of the prismatic specimen is marked with l, while the elongated (or shortened) length is marked with L. The relative elongation is interpreted with the relationship $\varepsilon_l = (L - l)/l$; (see figure 2). The thus interpreted specific elongation locally describes the strain of the body. This strain is also called linear strain. The linear strain can also be interpreted by comparing the elongation of the specimen with the length of the elongated (shortened) length, $\varepsilon_l = (L - l)/L$, rather than the length of the specimen. It can be shown that as long as with the ε_l is negligible to compare with 1 (one), then the two interpretations equal with errors negligible to compare with 1 (one). It also means that the above interpretation of linear strain has a limit: the linear strains cannot be large. This does not significantly affect the interpretation of strain, since the large-scale change of shape is not a strain but a rearrangement [15].

The length of the cylindrical specimen is marked with l, its radius with r, the rotation of two end cross-sections compared to each other, marked with Φ, the relative angular change is interpreted with the relationship: $\varepsilon_\varphi = (r\Phi)/l$ (see figure 2). The thus interpreted specific angular change locally describes the strain of the body. This strain is also called for change. The angle change strain can also be interpreted as comparing the angle change of the specimen with the length of the elongated (shortened) length or the length of the twisted line, $\varepsilon_\varphi = (r\Phi)/L$. It can be shown that as long as with ε_φ is negligible to compare with 1 (one), then the two interpretations equal each to other with errors negligible to compare with 1 (one). It also means that the above interpretation of form change has a limit: the angular changes cannot be large. Regarding this, see the note made at the linear strain.

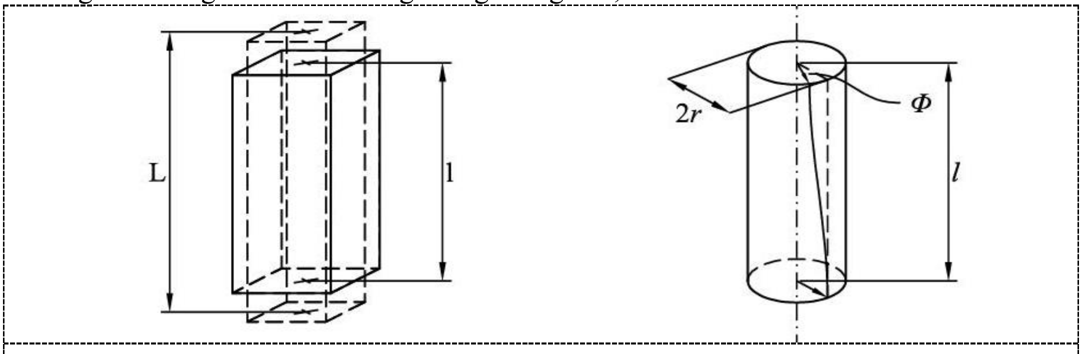

Figure 2 Experimental interpretation of strain: relative elongation and angle change

4.2. Mathematical interpretation of strain

The experimental interpretation of strain refers to a homogeneous strain state. This is not a general strain state. Therefore, in the case of arbitrary strain, it is not possible to characterize the strain of the body by the relative elongation of a particular selected curve. Analyzing the specific elongation of the specimen, we interpret the relative elongation of a (regular) arc (curve section) within a body: $\varepsilon_l = (L-l)/l,$ where l is the length of the selected (regular) arc before the deformation, and L marks the length following the deformation. Then the strain of a body can be interpreted: the strain of the body is the set of the relatíve elongation of all regular arces (curve sections) in the volume of the examined body.

The strain is based on the relative elongation of the curve sections and does not formally say anything about the change of angle closed by the curves. In terms of a triangle, the relationship between elongation and angular change can be interpreted.

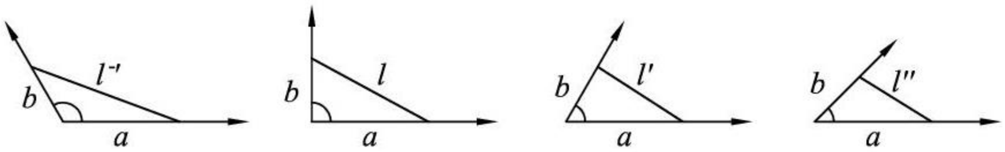

Figure 3 For the interpretation of the relationship between angular change and elongation

Consider the triangles shown in figure 3. The line segments *a* and *b* can lock different angles. Changes in angles are characterized by relative elongation applying to different lengths *l*. If the relative elongation of line segments *a* and *b* is negligible, i.e. negligible to compare with 1 (one), then the angle change is characterized by the relative elongation of the length *l* of the section stretched by the two line segments with errors negligible to compare with 1 (one).

4.3. Small and large strain

The interpretation of strain allows the strains to be grouped according to the scale of the strains.

The strain of a body is small if the relative elongation of any (regular) arc connecting any two points of the body within the body is sufficiently small, i.e. less than a negligible to compare with 1 (one) value.

The strain of a body is great if there is at least one regular arc that connects any two points of the body within the body with a relatively not small elongation.

5. Expressing strain with the change of the metric tensor

The strains described above characterize the strain of the body globally (in the integral form). For local characterization, arbitrary regular arcs should be located at every point of the body, with three fixed-length regular arcs – the base vectors as infinitely short arcs. In this case, it should be pointed out that the relative elongation in the three directions and relative angular changes of the three directions in one point already determine the relative elongation of the basic vector in any direction at the same point, and the relative angular change between any two basic vectors at the same point. If the relative elongations and relative angular change constitute a tensor quantity, they clearly determinate these strains.

5.1. Geometric content of metric tensor elements

The elements of the metric tensor are partly related to the relative elongation of the base vectors and partly to the angle between the base vectors (see e.g. [9,10,2]). Relative elongation of the base vectors can be expressed with the help of the main diagonal elements (there is no summary here and later according to the repeated indices)

$$\eta_{ii} = \frac{\sqrt{G_{ii}} - \sqrt{g_{ii}}}{\sqrt{g_{ii}}}, \tag{10}$$

and its angular change

$$\eta_{ij} = \alpha_{ij} - A_{ij}, \tag{11}$$

where

$$\cos\alpha_{ij} = \frac{g_{ij}}{\sqrt{g_{ii}g_{jj}}} \quad \text{and} \quad \cos A_{ij} = \frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}}. \tag{12}$$

Note that indexing applying to the η quantity suggests that we are dealing with a secondary rank tensor; if it is true, it would have to prove it. The terms (10-12) above show that the nine η_{ij} amounts do not form a tensor.

5.2. Expressing small strain with metric tensor change

The explication below is based on our previous results [3,4,5].

Relationships (10-12) do not only mean that the nine η_{ij} quantities do not form a tensor, but it can also be seen that the η_{ij} quantities depend on components of the measure tensor of strain $\boldsymbol{\gamma} = \mathbf{G} - \mathbf{g}$ in a complex way.

Let's transform the (10) relationship!

$$\eta_{ii} = \frac{\sqrt{G_{ii}} - \sqrt{g_{ii}}}{\sqrt{g_{ii}}} = \sqrt{1 + \frac{\gamma_{ii}}{g_{ii}}} - 1. \tag{13}$$

We can have two assumptions. *The first assumption* is that the relative elongations, i.e., η_{ii} quantities, are considered to be small, that is negligible to compare with 1 (one). Then the square root expression can be expressed in a series and in it it is enough to keep some elements. At the same time, this also means that the main diagonal elements of the measure tensor of strain, γ_{ii}, are negligible in relation to the main diagonal elements of the metric tensor of the undeformed state, g_{ii}. *The second assumption* is to express the square root expression in a series, and some first elements approximate the square root with sufficient accuracy. This means, on the one hand, that the main diagonal elements of the measure tensor of strain, γ_{ii}, are negligible in relation to the main diagonal elements of the metric tensor of the undeformed position, g_{ii}, on the other hand, that the η_{ii} quantities are small and negligible to compare with 1 (one). The two assumptions are equivalent.

By series expansion of the radical expression and retained by the two elements (13) it can formulate as follows:

$$\eta_{ii} = \sqrt{1+\frac{\gamma_{ii}}{g_{ii}}} - 1 \cong \frac{\gamma_{ii}}{2g_{ii}} - \frac{1}{2}\left(\frac{\gamma_{ii}}{2g_{ii}}\right)^2 = \frac{\gamma_{ii}}{2g_{ii}}\left(1 - \frac{1}{4}\frac{\gamma_{ii}}{g_{ii}}\right). \tag{14}$$

Since the condition of series expansion is that the quotient of γ_{ii}/g_{ii} in other words, the relative elongation at 1 (one) is negligible, the relative elongation in the case of small relative elongations can be therefore described with the expression

$$\eta_{ii} = \frac{\gamma_{ii}}{2g_{ii}}\left(= \frac{1}{2}\frac{\gamma_{ii}}{\sqrt{g_{ii}}\sqrt{g_{ii}}}\right). \tag{15}$$

For the "explication" of the expression (11), consider the $A_{ij} = \alpha_{ij} - \varphi_{ij}$ expression, where φ_{ij} marks the angular change of the i^{th} and the j^{th} base vectors. Then the following relationship can be described:

$$\cos A_{ij} = \cos\left(a_{ij} - \varphi_{ij}\right) = \cos a_{ij} \cos\varphi_{ij} + \sin a_{ij} \sin\varphi_{ij}. \tag{16}$$

The relationship means that the correlation in the case of a rectangular coordinate system is greatly simplified.

The case of orthogonal coordinates

The relationship (16) is simplified to the relationship

$$\cos A_{ij} = \sin\varphi_{ij}. \tag{17}$$

From this, the function of the angle change φ_{ij} can be expressed as a first two member of polynomial series if the angle change itself is negligible at 90°, i.e. if the relative (and thus the absolute) angle change is small. You can then write instead (17):

$$\varphi_{ij} = \cos A_{ij} = \frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}}. \tag{18}$$

Using repeatedly that the main diagonal elements of the measure tensor, γ_{ii}, are negligible to compare with main diagonal elements of the metric tensor of the undeformed state, g_{ii},, in the case of orthogonal coordinates, for the change of the angles of the base vectors we obtain the expression:

$$\varphi_{ij} = \frac{\gamma_{ij}}{\sqrt{g_{ii}g_{jj}}}, \quad (i \neq j). \tag{19}$$

Except for coefficient ½, this term is the same as that of the small relative elongation below (15).

The case of oblique coordinates

Assuming in expression (16), that the φ_{ij} function of the angle change is negligible to compare with 90°, and applying the series expansion for the cos and sin angle functions up to the quadratic elements ($\cos\varphi_{ij} \cong 1 - \varphi_{ij}^2/2$ and $\sin\varphi_{ij} \cong \varphi_{ij}$) the (16) expression can be transformed into a quadratic function the roots of which:

$$\varphi_{ij} = \operatorname{tg}\alpha_{ij}\left(1 \pm \sqrt{1 - 2\frac{\cos\alpha_{ij}}{\sin\alpha_{ij}}\frac{\cos A_{ij} - \cos\alpha_{ij}}{\sin\alpha_{ij}}}\right), \quad (i \neq j). \tag{20}$$

Small angle change only results in a solution with a negative sign. We apply series expansion again, and the elements of the series expansion, up to the quadratic elements, give the following expression to describe the angle change:

$$\varphi_{ij} = \frac{\cos A_{ij} - \cos\alpha_{ij}}{\sin\alpha_{ij}} + \frac{1}{2}\frac{\cos\alpha_{ij}}{\sin\alpha_{ij}}\left(\frac{\cos A_{ij} - \cos\alpha_{ij}}{\sin\alpha_{ij}}\right)^2, \quad (i \neq j). \tag{21}$$

(To retain cubic or higher power elements, the term (20) cannot be used.)

To examine the term (21), first consider the second members. The member in front of the parentheses in the near-right coordinate system gives a value close to zero, and if the first member is negligible to compare with 1 (one), then the second member is negligible to compare with the first member.

Therefore, the angle change can be expressed in an oblique coordinate system as:

$$\varphi_{ij} = \frac{\cos A_{ij} - \cos\alpha_{ij}}{\sin\alpha_{ij}} = \frac{\frac{G_{ij}}{\sqrt{G_{ii}}\sqrt{G_{jj}}} - \frac{g_{ij}}{\sqrt{g_{ii}}\sqrt{g_{jj}}}}{\sqrt{1 - \frac{g_{ij}^2}{g_{ii}g_{jj}}}}, \quad (i \neq j). \tag{22}$$

Using again the main diagonal elements of the measure tensor of strain, γ_{ii}, are negligible to compare with the metric tensor elements of the undeformed state, g_{ii}, in (22) we get:

$$\varphi_{ij} = \frac{\gamma_{ij}}{\sqrt{g_{ii}g_{jj} - g_{ij}^2}}, \quad (i \neq j). \tag{23}$$

In an oblique coordinate system, the relationship (23) only equals with the (19) relationship with errors negligible to compare with 1 (one), if the coordinate system is close to orthogonal one, i.e. the base vectors close an angle between 85 to 95 degrees.

In the case of small strains, i.e. small relative elongations and small relative angular changes, and in a slightly oblique coordinate system, we found a relationship between the specific elongation, the specific angular change and the measure tensor of strain:

$$\eta_{ii} = \frac{\gamma_{ii}}{2\sqrt{g_{ii}}\sqrt{g_{ii}}}, \tag{24}$$

$$\eta_{ij} = \varphi_{ij} = \frac{\gamma_{ij}}{\sqrt{g_{ii}g_{jj} - g_{ij}^2}}, \quad (i \neq j). \tag{25}$$

In accordance with this, the tensor of small strains can be introduced.

We call to the tensor of *small* strains the tensor interpreted with physical components:

$$\tilde{\varepsilon}_{ij} = \frac{\gamma_{ij}}{2\sqrt{g_{ij}}\sqrt{g_{ij}}}. \tag{26}$$

The covariant components of small strains in the base $\{\mathbf{R}_k\}$ in the deformed state are:

$$\varepsilon_{ij} = \frac{\gamma_{ij}}{2}. \tag{27}$$

5.3. Conditions of introducing the tensor of strain and its consequences

We introduced the tensor of strain in such a way, that the tensor of strain let have a direct physical meaning: the main diagonal elements of strain tensor give the relative elongation of the tangent vectors of the coordinate lines, and the side diagonal elements of strain tensor gives the change of the angles between the coordinate lines (see e.g. [2]).

The conditions for introducing the strain tensor are as follows: relative elongation and relative angular changes are small; the coordinate system used to describe the problem is orthogonal at least nearly orthogonal.

The consequences of introducing the strain tensor can be summarized as follows. The deformations can only be small. During the derivation of formula, the small magnitude of relative elongation and relative angular change were used several times from the radical and trigonometric expressions. For this reason, it is difficult to assume that other strain tensors of large deformation could be derived by using a metric tensor-based expression of relative elongation and relative angular change from the above relationships. In pragmatic terms: there is *one strain tensor* and it only *applies to small strains*; *tensor of large strains cannot be derived*. This is not in contradiction with the physically experienced phenomena: the phenomenon considered to be a large strain is not deformation, but a rearrangement, for which the concept of the coordinate line cannot be interpreted, since the rearrangement of the particles excludes the topological order from being preserved [15]. *The strain tensor is symmetrical*. The side diagonal elements define the relative angle change between two vectors. Since the definition of the angle from one vector to the other or the angle from the second to the first is always the same, an asymmetric metric tensor cannot be formed. For the interpretation of an asymmetric metric tensor the interpretation of the metric tensor should be fundamentally changed. For this reason, the interpretation of an asymmetric strain tensor requires mathematical foundations other than the Riemann geometry, such as changing the definition of a scalar product, or defining an angle between two lines different from the scalar product. (Knowing the structure of geometry and differential geometry makes it difficult to imagine a different mathematical construction for these mathematical objects.)

Due to the interpretation of the tensors of small strains, it is non-linear: the strain tensor was not derived from the gradient tensor of the displacement, but from the measure tensor of the strain interpreted as the difference of the metric tensor in the deformed and undeformed state. However, this is by definition a quadratic function of the gradient tensor $\nabla\mathbf{u}$ of the displacement vector $\mathbf{u}$. Based on the above, the linearization of nonlinear relationships of the tensor of small strains is not related to the magnitude of the strain but to something else [3,4,5].

6. Summary

In this study we provided an overview of applying metric tools and their limitations.

By metric tools we mean the scalar product, the interpretation and measurement of distance and angle.

The conditions for applying metric tools are the following:

- we model the media to be examined as a domain of a topological space,
- a symmetric quadratic form, the metric tensor is given in the topological space,
- the length and angle enclosed by the two curves are given by a scalar product,
- relative strains are negligible to compare with 1 (one),
- the applied coordinate system is orthogonal at least almost orthogonal.

The consequences of applying metric tools are the following.

- In the classic continuum independent rotational degree of freedom cannot be interpreted:
  - the rotation of a point cannot be interpreted,
  - rotation can only be interpreted of a direction, or in a rigid body-like neighborhood of a point (three points, or a triple),
  - the displacement in the classic continuum clearly determines the rotation of any base vector as a direction, and the rotation of the neighborhood of every point,
  - as a result of it, an angle change which is independent of the displacement in any sense cannot be interpreted,
  - since the metric tensor is symmetric (the elements of the scalar product are interchangeable), therefore, an asymmetric strain tensor, and, consequently, a rotation field independent of the derivative of the displacement vector cannot be interpreted.
- In the classic continuum

- interpretable a strain tensor with mechanical content the main diagonal elements describe the relative elongation and the side diagonal elements describe the relative angular change,
    - the strain tensor interpreted describes small strains,
    - the strain tensor describing large strains cannot be interpreted.
- Tensor of small strains
    - is half of the difference of the metric tensor of the deformed **G** and undeformed **g** state; and
    - is expressed, because of its interpretation, by the quadratic function of the gradient tensor of the displacement vector **u**: $\boldsymbol{\varepsilon} = (\nabla\mathbf{u} + (\nabla\mathbf{u})^{*} + (\nabla\mathbf{u})(\nabla\mathbf{u})^{*}) / 2$,
    - is symmetrical, and
    - linearization is independent of the magnitude of strain, more precisely, of that fact that the strains are small.
- The following conclusions can be regarding the rotations:
    - the rotation of a direction vector is given by the $\nabla\mathbf{R}$ gradient tensor of position vector **R**,
    - the rotation of the neighborhood of a point is given by the orthogonal tensor $\boldsymbol{\Omega}$ getting from the polar decomposition of the gradient tensor $\nabla\mathbf{R}$ of position vector **R** in the deformed state,
    - the rotation of the neighborhood of a point is described in the first, linear approach by the tensor $\boldsymbol{\Omega}_L = \mathbf{g} + \boldsymbol{\omega}_L$,
    - the tensor $\boldsymbol{\omega}_L = ((\nabla\mathbf{u})^{*} - \nabla\mathbf{u}) / 2$ does not describe the rotation, but instead interprets a vector product.

The tensor of small strains is consistent with the atomic-molecular structure and continuous mapping. The fact that a tensor of large strains cannot be created does not contradict the experience, since the significant change of shape is not a strain but a rearrangement.

References

[1] Goldenblat I I 1969 *Some Problems of Theory of Elasticity* (Moscow: Nauka) (In Russian)
[2] Lurie A I 2005 *Theory of Elasticity* (Berlin: Springer-Verlag)
[3] Lámer G 1985 Notes on the theory of large displacement with small strain *Periodica Politechnica* vol 29 (1-2) pp. 53-65
[4] Lámer G 1990 *Mathematical Foundation of the Theory of Perfectly Elastic Shells and Rods Underdoing Large Displacement with Small Strain* Thesis. (Budapest) (In Hungarian)
[5] Lámer G 1992-93 On the kinematics of continuum underdoing large displacement with small strain *Építés-, Építészettudomány* vol XXIII (1-2) pp. 35-59 (In Hungarian)
[6] Kobayashi S and Nomizu K 1963-69 *Foundations of Differential Geometry* vol I-II. (New York: Interscience Publisher)
[7] Korn G A and Korn T M 2000 Math*ematical Handbook for Scientists and Engineers. Definitions, Theorems, and Formulas for Reference and Review.* (New York: Dover Publications)
[8] Levi-Civita T 1926 *The Absolute Differential Calculus* (London: Blackie & Son Limited)
[9] Green A E and Zerna W S 1954 *Theory of Elasticity* (Oxford: The Clarendon Press)
[10] Love A E H 1927 *A Treatise on the Mathematical Theory of Elasticity* 4th ed. (Cambridge, At the University Press)
[11] Appell P 1928 *Traité de Mécanique Rationelle*. Tome Troisiéme. 3. Ed. (Paris: Gauther-Villars)
[12] Appell P 1926 *Traité de Mécanique Rationelle*. Tome Cinquieme. (Paris: Gauther-Villars)
[13] Budó Á 1972 *Mechanics*. 5th Ed. (Budapest: Tankönyvkiadó) (In Hungarian)
[14] Rózsa, P 1974 *Linear Algebra* (Budapest: Műszaki Könyvkiadó) (In Hungarian)
[15] Lámer G Notes on the possibilities and limitations of using topological tools in mechanical modeling of media (*In this Issue*)